

We used \underline{u} for fluid velocity, but I want to change to use \underline{v} instead \rightarrow a new field for every particle.

$$\nabla \cdot \underline{v} \cong \sum_b \frac{m_b}{\rho_b} \underline{v}_b \cdot \nabla W(|\underline{r} - \underline{r}_b|, h)$$

↑ Monahan cubic spline
↑ size of the kernel

$$\begin{aligned} \nabla \cdot \underline{v} &\equiv \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot (v_x \hat{i} + v_y \hat{j} + v_z \hat{k}) \\ &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \end{aligned}$$

$$\nabla \cdot \underline{v} = \frac{1}{\rho} [\nabla \cdot (\rho \underline{v}) - \underline{v} \cdot \nabla \rho]$$

$$\nabla \cdot \underline{v}_a \cong \frac{1}{\rho_a} \left[\sum_b m_b \frac{\rho_b \underline{v}_b}{\rho_b} \cdot \nabla_a W_{ab} - \underline{v}_a \cdot \sum_b m_b \nabla_a W_{ab} \right]$$

$$\rho_a \nabla \cdot \underline{v}_a \cong \sum_b m_b (\underline{v}_b - \underline{v}_a) \cdot \nabla_a W_{ab}$$

↑ switch $a \leftrightarrow b$
 $W(|\underline{r}_a - \underline{r}_b|, h)$

$$\frac{de}{dt} = -\frac{P}{\rho} \nabla \cdot \underline{v}$$

$$\frac{de_a}{dt} = \left(\frac{\rho_a}{\rho_a^2} \right) \sum_b m_b (\underline{v}_a - \underline{v}_b) \cdot \nabla_a W_{ab}$$

Benz Formulation

continuity

$$\rho_a = \sum_b m_b W_{ab}$$

Momentum equation

$$\frac{d\underline{v}_a}{dt} = - \frac{\rho_a \nabla P_a}{\rho_a^2}$$

$$\frac{d\underline{v}_a}{dt} = - \frac{1}{\rho_a^2} \sum_b m_b (P_b - P_a) \nabla_a W_{ab}$$

$$F_{ab} = -F_{ba}$$

Force = 0 for constant Pressure, but it does not conserve momentum

$$F_{ab} = -F_{ba}$$

Force = 0 for constant pressure,
but it does not conserve momentum
nor angular momentum.

Instead:
$$\frac{\nabla P}{\rho} = \nabla\left(\frac{P}{\rho}\right) + \frac{P}{\rho^2} \nabla \rho$$

$$\frac{d\underline{v}_a}{dt} = - \sum_b m_b \left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2} \right) \nabla_a W_{ab}(r_{ab}, h)$$

$$a \leftrightarrow b \quad \nabla_a \rightarrow -\nabla_b$$

Newton's 3rd Law $F_{ab} = -F_{ba}$

same for
a & b?

How to conserve momentum when particles
have different kernel sizes h_a, h_b ?

Symmetrizing the Kernel

$$\overset{a}{\bullet} \longleftrightarrow \overset{b}{\bullet} \quad \left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2} \right) \checkmark$$

$$W_{ab}(r_{ab}, \underline{h}_a)$$

$$h_{ab} = \frac{1}{2}(h_a + h_b) \checkmark$$

nice, but difficult to
implement

$$W_{ab} = \frac{1}{2} \left(W(r_{ab}, h_a) + W(r_{ab}, h_b) \right) \checkmark$$

"easy" to implement.

$$\textcircled{1} \quad \frac{d\underline{v}_a}{dt} += -\frac{1}{2} \left[\sum_b m_b \underbrace{F_{ab}}_{\left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2} \right)} \nabla_a W(r_{ab}, h_a) \right] \text{ "gather"}$$

$$\textcircled{2} \quad \frac{d\underline{v}_a}{dt} += -\frac{1}{2} \sum_b m_b F_{ab} \nabla_a W(r_{ab}, h_b) \text{ "gather"}$$

rewrite ② swapping $a \leftrightarrow b$

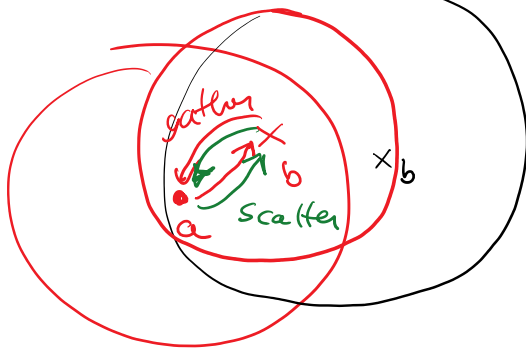
$$\frac{d\underline{v}_b}{dt} += -\frac{1}{2} \sum_a m_a F_{ab} \nabla_b W(r_{ab}, h_a)$$

$$\frac{d\mathbf{v}_b}{dt} = -\frac{1}{2} \sum_a m_a F_{ab} \nabla_b W(r_{ab}, h_a)$$

but $\nabla_b = -\nabla_a$

$$\frac{d\mathbf{v}_b}{dt} = \frac{1}{2} \sum_a m_a \boxed{F_{ab} \nabla_a W(r_{ab}, h_a)}$$

"scatter"



Only need to calculate the contribution in once for each neighbor

Gather scatter algorithm.

Remember to initialize $\frac{d\mathbf{v}_a}{dt} = 0$ for all particles.



Navier-Stokes Equations
Shock "Direct" Numerical

Add Artificial Viscosity

$$F_{ab} = \frac{p_a}{\rho_a^2} + \frac{p_b}{\rho_b^2} + \Pi_{ab}$$

$$\Pi_{ab} = \begin{cases} \frac{-\alpha \bar{c}_{ab} N_{ab} + \beta N_{ab}^2}{\bar{\rho}_{ab}}, & \mathbf{v}_{ab} \cdot \mathbf{r}_{ab} < 0 \\ 0, & \mathbf{v}_{ab} \cdot \mathbf{r}_{ab} \geq 0 \end{cases}$$

$$\bar{c}_{ab} = \frac{1}{2}(c_a + c_b)$$

$$\bar{\rho}_{ab} = \frac{1}{2}(\rho_a + \rho_b)$$

$$N_{ab} = \frac{\bar{h}_{ab} \mathbf{v}_{ab} \cdot \mathbf{r}_{ab}}{r_{ab}^2 + \eta^2}$$

$$\beta = 2\alpha$$

$$\alpha = 1 \quad \beta = 2$$

$$\alpha = 0.5 \quad \beta = 1$$

$$\bar{h}_{ab} = \frac{1}{2}(h_a + h_b)$$

small number

reasonable

to keep a reasonable denominator

$$e_a = \frac{P_a}{\rho_a(\gamma-1)}$$

$$c_a = \sqrt{\gamma(\gamma-1)e_a}$$

$$c_a = \sqrt{\gamma \frac{P_a}{\rho_a}}$$

$$c_a^2 = \gamma \frac{P_a}{\rho_a}$$

P \leftarrow this way

$\gamma \rho$

Need c

$$\frac{P_a}{\rho_a^2} = \frac{c_a^2}{\gamma \rho_a}$$

Let's write the code ...

Variables for particles: $\underline{r}, \underline{v}, e, c, \rho, h$

Need to calculate: $\underline{a} = \frac{d\underline{v}}{dt}$ $\dot{e} = \frac{de}{dt}$

★ Need 2 extra temporary variables \rightarrow ODE integration using a Leapfrog-like algorithm.

$\underline{v}_{pred}, e_{pred}$

Functions: DRIFT1(), DRIFT2()
KICK(), CALCFORCE()

SPH:

DRIFT1($\Delta t=0$)
CALCFORCE()

Bootstrap step for $\underline{v}_{pred}, e_{pred}$

for (step=0; step < NSTEPS; ++step) {

DRIFT1($\Delta t/2$)

CALCFORCE()

KICK(Δt)

DRIFT2($\Delta t/2$)

}

CALCFORCE:

TREEBUILD

NN-DENSITY

CALCSOUND

NN-SPHFORCE

all particles calculate ρ_a

all particles calculate c

All calculate \underline{a}, \dot{e}

TREE WALKS

DRIFT1(Δt):

$$\underline{\Gamma} += \underline{v} \Delta t$$

$$\underline{v}_{\text{pred}} = \underline{v} + \underline{a} \Delta t$$

$$\underline{e}_{\text{pred}} = \underline{e} + \dot{\underline{e}} \Delta t$$

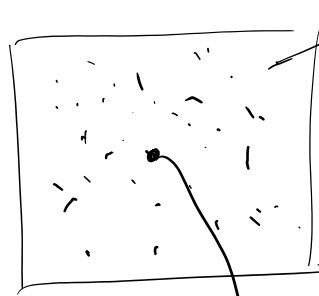
DRIFT2(Δt):

$$\underline{\Gamma} += \underline{v} \Delta t$$

KICK(Δt):

$$\underline{v} += \underline{a} \Delta t$$

$$\underline{e} += \dot{\underline{e}} \Delta t$$



$v = 0$
 $e = ?$ 10^{-5} $\Delta t = ?$

Periodic

$e = 1$ Sedov Blast wave