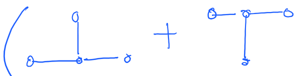
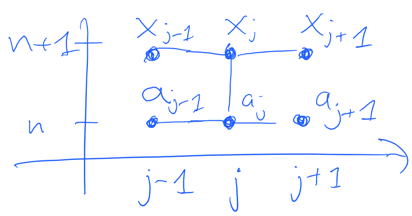
Schema 3: Crank-Nicholson

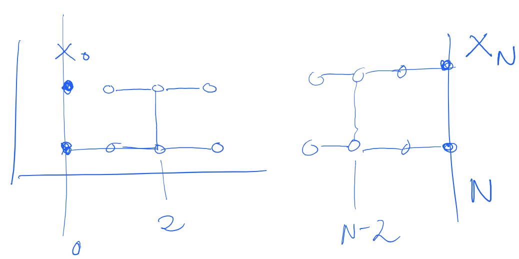
Mischt Schema 1 mit 2: $\frac{1}{2}\Big($  $\Big)$

$$u_j^{(n+1)} - u_j^{(n)} = \frac{\alpha}{2}\left(u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n)} + u_{j+1}^{(n+1)} - 2u_j^{(n+1)} + u_{j-1}^{(n+1)}\right)$$

Das ist jetzt 2-Ordnung in Zeit.



$$u_j^{n+1} \equiv x_j$$
$$u_j^{n} \equiv a_j$$



x_0 und x_N sind auch noch gegeben!

$$\underline{\underline{A}}\,\underline{x} = \underline{b}$$

Die Gittergleichungen sind dann:

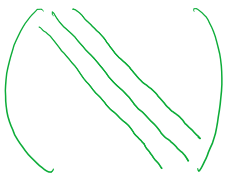
$j=2..N-2$ $\quad -\frac{\alpha}{2}x_{j-1} + (1+\alpha)x_j - \frac{\alpha}{2}x_{j+1} = \frac{\alpha}{2}a_{j-1} + (1-\alpha)a_j + \frac{\alpha}{2}a_{j+1}$

$j=1$ $\quad (1+\alpha)x_1 - \frac{\alpha}{2}x_2 = \frac{\alpha}{2}(x_0 + a_0 + a_2) + (1-\alpha)a_1$

$j=N-1$ $\quad -\frac{\alpha}{2}x_{N-2} + (1+\alpha)x_{N-1} = \frac{\alpha}{2}(a_{N-2} + a_N + x_N) + (1-\alpha)a_{N-1}$

$$\begin{pmatrix} (1+\alpha) & -\frac{\alpha}{2} & \emptyset & \emptyset & \emptyset & \cdots & \emptyset \\ -\frac{\alpha}{2} & (1+\alpha) & -\frac{\alpha}{2} & & \emptyset & & \\ \emptyset & -\frac{\alpha}{2} & (1+\alpha) & -\frac{\alpha}{2} & & & \\ & & & \ddots & & & \\ \emptyset & \emptyset & & \cdots & \emptyset & -\frac{\alpha}{2} & (1+\alpha) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-2} \\ x_{N-1} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \\ b_{N-1} \end{pmatrix}$$

$$\begin{pmatrix} & & & & & \\ \emptyset & \emptyset & \cdots & \emptyset & -\frac{\alpha}{2} & (1+\alpha) \end{pmatrix} \begin{pmatrix} \vdots \\ x_{N-2} \\ x_{N-1} \end{pmatrix} = \begin{pmatrix} \vdots \\ b_{N-1} \end{pmatrix}$$

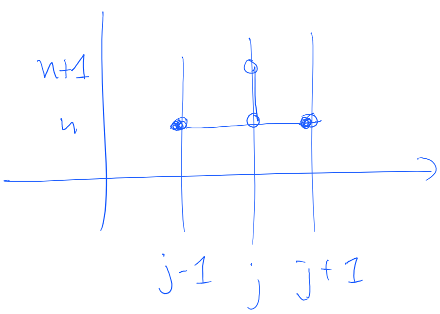


Tridiagonal $2N$

Lösung in: $O(N)$

„Hydrodymik" Einfache Hyperbolische Beispiele

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \quad \text{„Advection Equation"}$$

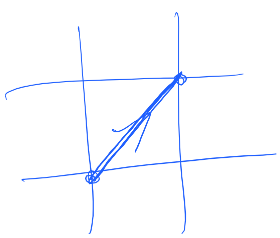


$$\frac{\partial u}{\partial t} \approx \frac{u_j^{(n+1)} - u_j^{(n)}}{\Delta t}$$

$$\frac{\partial u}{\partial x} \approx \frac{u_{j+1}^{(n)} - u_{j-1}^{(n)}}{2 \Delta x}$$

$$u_j^{(n+1)} = u_j^{(n)} - \frac{1}{2} C \left(u_{j+1}^{(n)} - u_{j-1}^{(n)} \right) \qquad C := \frac{\Delta t\, a}{\Delta x}$$

Courant Zahl



Das Verhältnis zwischen Wellengeschwindigkeit und der „Gittergeschwindigkeit".

Stabilitätsanalyse: $u_j^{(n)} = \xi^n e^{ij\theta}$ $\quad i = \sqrt{-1}$

$$A^{n+1} e^{ij\theta} = A^n e^{ij\theta} - \frac{1}{2} C A^n \left(e^{i(j+1)\theta} - e^{i(j-1)\theta} \right)$$

Kreuzen

$$A = 1 - C \left(\frac{e^{i\theta} - e^{-i\theta}}{2} \right)$$

$$A = 1 - C\left(\frac{\quad}{2}\right)$$

$$\boxed{A = 1 - i c \sin\Theta}$$

Für Stabilität $\|A\| < 1$

$$\|A\| = 1 + c^2 \sin^2\Theta > 1 \; !!$$

USELESS!

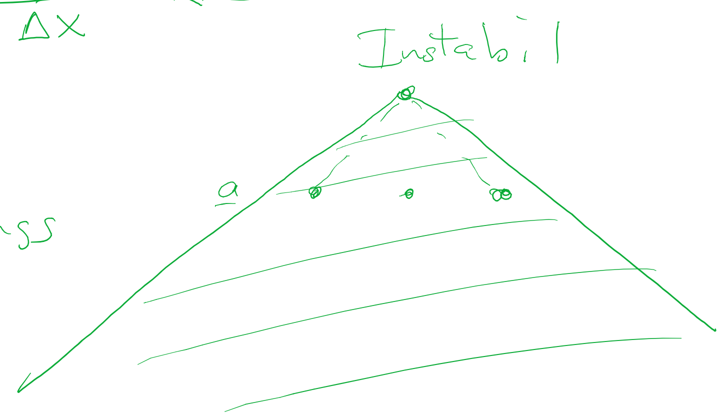
LAX Methode: $u_j^{(n)} \longrightarrow \frac{1}{2}\left(u_{j+1}^{(n)} + u_{j-1}^{(n)}\right)$

Anstatt

$$u_j^{(n+1)} = \frac{1}{2}\left(u_{j+1}^{(n)} + u_{j-1}^{(n)}\right) - \frac{1}{2} c \left(u_{j+1}^{(n)} - u_{j-1}^{(n)}\right)$$

$$\Rightarrow A = \cos\Theta - i c \sin\Theta$$

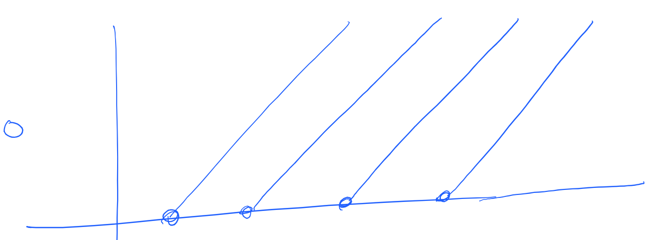
Stabil $\Rightarrow \dfrac{|a|\,\Delta t}{\Delta x} \leq 1$



Upwind Methoden

a ist +ve

$$\frac{\partial u}{\partial x} \simeq \frac{u_j^{(n)} - u_{j-1}^{(n)}}{\Delta x} \qquad a > 0$$



CIR $\quad u_{j+1}^{(n)} - u_j^{(n)} \qquad a < 0$

$$u(t,x) = u(0, x - at)$$

CIR

Upwind

$$\frac{u_{j+1}^{(n)} - u_j^{(n)}}{\Delta x} \quad a < 0 \qquad u(t,x) = u(0, x-at)$$

$$\|A\|^2 < 1 \quad \text{wenn} \quad 0 \le c < 1$$

$$u_j^{(n+1)} = u_j^{(n)} - c\left(u_j^{(n)} - u_{j-1}^{(n)}\right) \quad \text{für } a > 0$$

LAX - Wendroff 2^{te} Ordnung in Raum & Zeit

$$u_j^{(n+1)} = \frac{1}{2} c(1+c)\, u_{j-1}^{(n)} + (1-c^2)\, u_j^{(n)} - \frac{1}{2} c(1-c)\, u_{j+1}^{(n)}$$