Finite Volume Method: 1-D and 2-D

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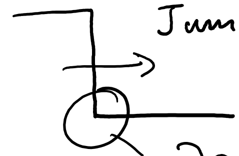
Finite Difference to approximate the derivatives: $\frac{\partial \rho}{\partial t} + a \frac{\partial \rho}{\partial x} = 0$

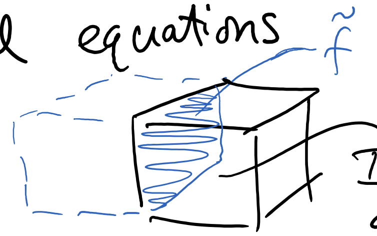
Conservation of Mass:



Jump Shock-Wave propagating

$\frac{\partial \rho}{\partial x}$ is not good here!

Integral equations

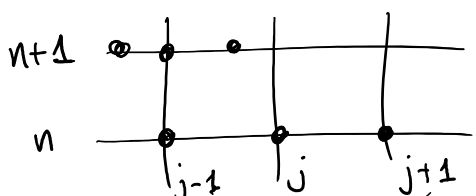


Integrate over a volume

$$\rho_j^{n+1} = \rho_j^n + \frac{\Delta t}{\Delta x}\left[\underbrace{f_{j-\frac{1}{2}} - f_{j+\frac{1}{2}}}_{\text{Approximated numerically}}\right]$$

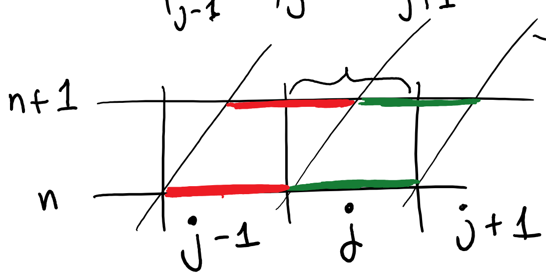
Integration over all fluxes should be exact.

For linear advection $f(\rho) = a \cdot \rho$, $a \geq 0$



Finite Difference

characteristics with a slope of a



Finite Volume

$$\rho_j^{n+1} = \underbrace{\boxed{\left(\frac{a \cdot \Delta t}{\Delta x}\right)}}_{C} \rho_{j-1}^n + \underbrace{\boxed{\left(1 - \frac{a \cdot \Delta t}{\Delta x}\right)}}_{(1-C)} \rho_j^n$$

Godunov Method

$$\rho_j^{n+1} = C \rho_{j-1}^n + (1-C)\, \rho_j^n$$

$$\rho_j^{n+1} - \rho_j^n + C\left(\rho_j^n - \rho_{j-1}^n\right) = 0$$

$$\rho_j^{n+1} - \rho_j^n + C\left(\rho_j^n - \rho_{j-1}^n\right) = 0$$

1st order Upwind Scheme
"CIR Method"

Results in the same as for Finite Difference, in this case.

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0$$

Thing that gets added to this equation is a diffusive term.

$\frac{\partial^2 \rho}{\partial x^2}$ Numerical diffusion gets added.

$$\frac{\rho_j^{n+1} - \boxed{\rho_j^n}}{\Delta t} + a\,\frac{\rho_{j+1}^n - \rho_{j-1}^n}{2\,\Delta x} = 0$$

Not good right!

Taylor expand ρ in time to 2nd order:

plug in

$$\rho_j^{n+1} = \boxed{\rho_j^n} + \Delta t\left(\frac{\partial \rho}{\partial t}\right) + \frac{\Delta t^2}{2}\left(\frac{\partial^2 \rho}{\partial t^2}\right) + \dots$$

Taylor expand ρ in space to 2nd order

$$\rho_{j+1}^n = \rho_j + \Delta x\left(\frac{\partial \rho}{\partial x}\right) + \frac{\Delta x^2}{2}\left(\frac{\partial^2 \rho}{\partial x^2}\right) + \dots$$

$$\rho_{j-1}^n = \rho_j - \Delta x\left(\frac{\partial \rho}{\partial x}\right) + \frac{\Delta x^2}{2}\left(\frac{\partial^2 \rho}{\partial x^2}\right) + \dots$$

2x cancels

$$\frac{\cancel{\Delta t}\left(\frac{\partial \rho}{\partial t}\right) + \frac{\Delta t^2}{2}\left(\frac{\partial^2 \rho}{\partial t^2}\right)}{\cancel{\Delta t}} + a\,\frac{\left(\cancel{2\Delta x}\left(\frac{\partial \rho}{\partial x}\right)\right)}{\cancel{2\Delta x}} = 0$$

$\cancel{\Delta t}$

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = \boxed{-\frac{\Delta t}{2}\left(\frac{\partial^2 \rho}{\partial t^2}\right)} + \mathcal{O}(\Delta t^2, \Delta x^2)$$

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial^2 \rho}{\partial t^2} + a\frac{\partial}{\partial x}\Big(\overbrace{\frac{\partial \rho}{\partial t}}^{-a\frac{\partial \rho}{\partial x}}\Big) = 0$$

$$\frac{\partial^2 \rho}{\partial t^2} = -a\frac{\partial}{\partial x}\left(-a\frac{\partial \rho}{\partial x}\right)$$

$$\frac{\partial^2 \rho}{\partial t^2} = +a^2\frac{\partial^2 \rho}{\partial x^2}$$

$$\frac{\partial \rho}{\partial t} + \boxed{a\frac{\partial \rho}{\partial x}} = \underbrace{\boxed{-a^2\frac{\Delta t}{2}}}_{D}\left(\frac{\partial^2 \rho}{\partial x^2}\right)$$

Advection-Diffusion Equation

D is negative means it is unstable

Modified Equation

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = -a^2\frac{\Delta t}{2}\frac{\partial^2 \rho}{\partial x^2}$$

Look at this for the 1st order upwind method! What do you get?

2-D Advection

the velocity

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \vec{u}) = 0$$

$$\partial t + \nabla \cdot (\rho \underline{u}) = 0$$

$$\underline{u} = \langle a, b \rangle \qquad a, b > 0$$

$$\frac{\partial \rho}{\partial t} + a \frac{\partial \rho}{\partial x} + b \frac{\partial \rho}{\partial y} = 0$$

A first order finite difference (upwind):

$$\frac{\rho_{je}^{n+1} - \rho_{je}^{n}}{\Delta t} + a \frac{\boxed{\rho_{je}^{n}} - \boxed{\rho_{j-1\,e}^{n}}}{\Delta x} + b \frac{\rho_{je}^{n} - \boxed{\rho_{je-1}^{n}}}{\Delta y} = 0$$

Stability Analysis shows that:

$C_a > 0$ $\qquad \boxed{C_a = \frac{a \Delta t}{\Delta x}}$ $\qquad \boxed{\text{AND}}$ $\qquad \boxed{\frac{a \Delta t}{\Delta x} + \frac{b \Delta t}{\Delta y} \leq 0}$

$C_b > 0$

$$C_a + C_b \leq 1$$

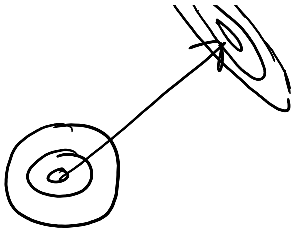
Sum of the Courant numbers in x and y must be less than 1.

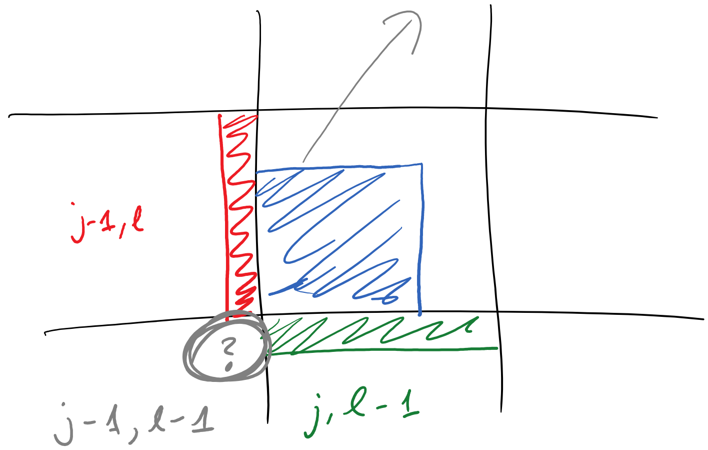
Modified equation is interesting:

$$\frac{\partial \rho}{\partial t} + a \frac{\partial \rho}{\partial x} + b \frac{\partial \rho}{\partial y} = \frac{a \Delta x}{2}(1 - C_a)\frac{\partial^2 \rho}{\partial x^2} + \frac{b \Delta y}{2}(1 - C_b)\frac{\partial^2 \rho}{\partial y^2} \boxed{- ab \Delta t \frac{\partial^2 \rho}{\partial x \partial y}}$$

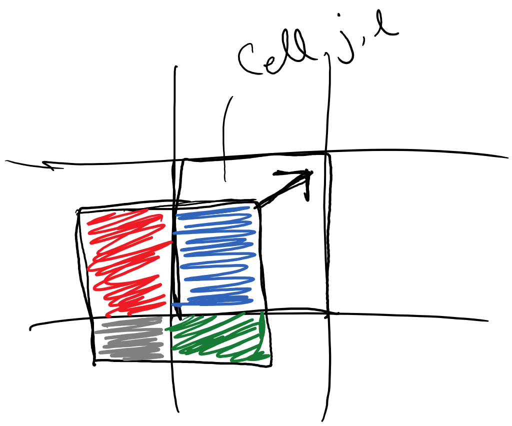
Won't preserve shape!

Diffusion in the tangential direction of motion and anti-diffusion in the direction of motion.





Corner Transport Upwind Method



$$\rho_{jl}^{n+1} = \underline{(1-C_a)(1-C_b)\rho_{jl}^{n}} + \underline{C_a(1-C_b)\rho_{j-1\,l}^{n}} + \underline{(1-C_a)\,C_b\,\rho_{j\,l-1}^{n}} + \underline{C_a C_b \rho_{j-1,\,l-1}^{n}}$$

Stability is (slightly) better:

$$0 \le C_a < 1 \qquad 0 \le C_b < 1$$

$$C_a = \frac{a\Delta t}{\Delta x} \qquad C_b = \frac{b\Delta t}{\Delta y}$$

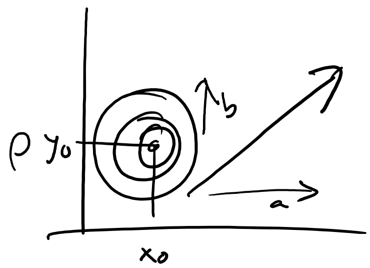
Rewrite as a 2-step method

$$\boxed{\begin{aligned} \rho_{jl}^{*} &= (1-C_a)\rho_{jl}^{n} + C_a \rho_{j-1\,l}^{n} \\ \rho_{jl}^{n+1} &= (1-C_b)\rho_{jl}^{*} + C_b \rho_{j\,l-1}^{*} \end{aligned}}$$

$$\boxed{\begin{aligned} \rho^{*}_{je} &= (1-C_a)\,\rho^{n}_{je} + C_a\,\rho^{n}_{j-1\,e} \\ \rho^{n+1}_{je} &= (1-C_b)\,\rho^{*}_{je} + C_b\,\rho^{*}_{j\,e-1} \end{aligned}}$$

Test 2 Methods in 2-D

1. CIR Finite Diff.
2. CTU Method



$$\rho(x,y) = A \exp\left[-\left(\frac{(x-x_0)^2}{2\sigma_x^2} + \frac{(y-y_0)^2}{2\sigma_y^2}\right)\right]$$

Initial Condition