Ordinary Differential Equations

12 October 2020 12:37

$$\int_{y_n}^{y_{n+1}} dy = \int_{x_n}^{x_{n+1}} f(x, y)\, dx$$

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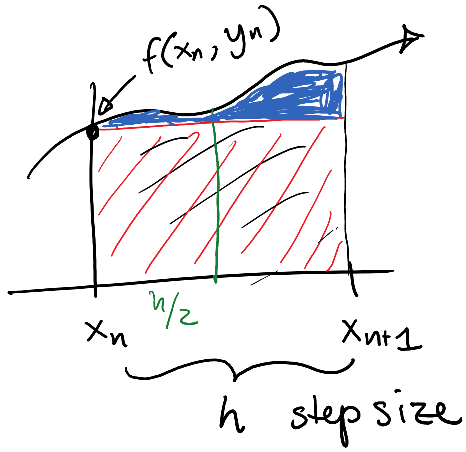
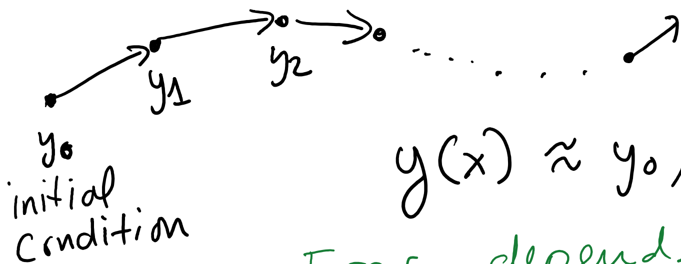


$$y_{n+1} - y_n = h \cdot f(x_n, y_n)$$

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$



$$y(x) \approx y_0, y_1, y_2, \ldots$$

Error depends on the choice of h, step size.

<u>Truncation Error</u>: Error associated with the algorithm or method, and not the precision of the floating point calculations.

Local Error : Error over one step

Global Error : Error over a fixed Interval

X : a global interval

$$\int_{x_n}^{x_{n+1}} f(x,y)\, dx \qquad f(x, y(x)) = f(x, \underbrace{y(x_n + h)}_{\text{Taylor expand}}))$$

$$\cong f(x, y_n + \underbrace{h \cdot \frac{dy}{dx}\Big|_{x_n}}_{\text{small}})$$

Taylor expand again

$$f(x, y(x)) \cong f(x, y_n)$$

$$+ h \cdot \Big[dy \Big|$$

$f(x,y(x)) = f(x,y_n)$ again

$+ h \cdot \boxed{\frac{dy}{dx}\Big|_{x_n}} f'(x,y_n)$

Slope at x_n is $f(x_n,y_n)$

$$\int_{x_n}^{x_{n+1}} f(x,y(x))\,dx \cong \int_{x_n}^{x_{n+1}} \left[f(x,y_n) + h f(x_n,y_n) f'(x,y_n)\right] dx$$

Fix the function values at our initial condition at x_n

$$\cong h \cdot f(x_n,y_n) + \boxed{h^2 f(x_n,y_n) f'(x_n,y_n)}$$

Local Error : $O(h^2)$

How many steps in the fixed interval?

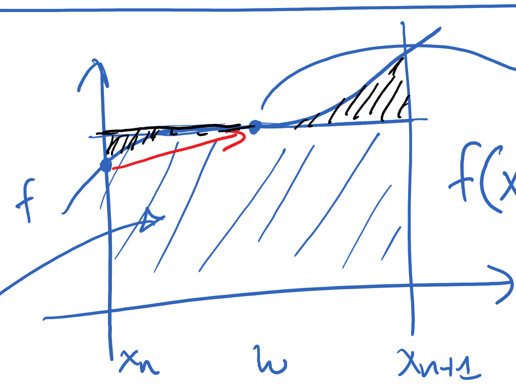
$$N_{steps} = \frac{X}{h}$$

Global error $\Rightarrow O(h)$

Quite Poor

$\boxed{\text{Forward Euler Method}}$

Can we do better?



$f(x_n + \frac{1}{2}, y_{n+1/2})$

$f(x_n + \frac{1}{2}h, y(x_n + \frac{1}{2}h))$

?
unknown

$h \cdot f(x_n + \frac{1}{2}h, y(x_n + \frac{1}{2}h))$

$$A = h \cdot f\left(x_n + \tfrac{1}{2}h,\ y(x_n + \tfrac{1}{2}h)\right)$$

Use Forward Euler to get this

$$y_{n+\frac{1}{2}} \cong y_n + \frac{h}{2} f(x_n, y_n)$$

$$\boxed{y_{n+1} - y_n = h \cdot f\left(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{h}{2} f(x_n, y_n)\right)}$$

2 evaluations of f here (so more expensive!)

Mid-point Runge-Kutta

Local error: $O(h^3)$

Global error: $O(h^2)$

4th-order Runge-Kutta

$$k_1 = h \cdot f(x_n, y_n)$$

$$k_2 = h \cdot f\left(x_n + \frac{h}{2},\ y_n + \frac{k_1}{2}\right)$$

$$k_3 = h \cdot f\left(x_n + \frac{h}{2},\ y_n + \frac{k_2}{2}\right)$$

$$k_4 = h \cdot f(x_n + h,\ y_n + k_3)$$

$$y_{n+1} = y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(h^5)$$

These are <u>explicit</u> methods.

Implicit Method: much more expensive

$$y_{n+1} = y_n + h \cdot f\left(\tfrac{1}{2}(x_n + x_{n+1}),\ \tfrac{1}{2}(y_n + y_{n+1})\right)$$

- Write a linear system
- iterate, start with a guess for y_{n+1}
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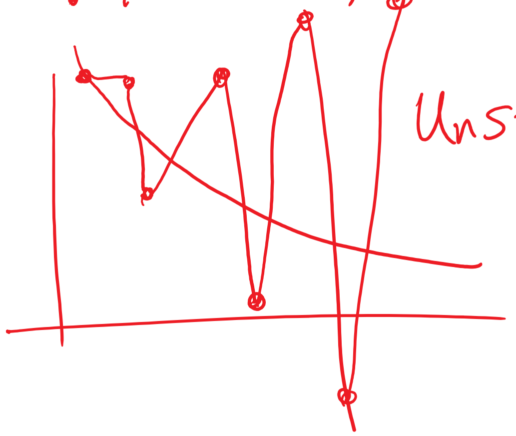
• iterate, start with a guess for y_{n+1}



STABILITY

Predator - Prey Behaviour

Foxes and Mice

f $\qquad$ m

Lotka - Voltera Model (1920)

without foxes: mice population grows without limitation.

$$\frac{\Delta m}{m} = k_m \cdot \Delta t$$

k_m ← constant Birthrate

But if foxes are around the population reduces proportional to the number of foxes.

$$\frac{\Delta m}{m} = k_m \cdot \Delta t - k_{mf} \cdot f \cdot \Delta t$$

$$\frac{dm}{dt} \approx \frac{\Delta m}{\Delta t} = (k_m \cdot m - \underbrace{k_{mf} \cdot m \cdot f}_{\propto \text{ number of encounters}})$$

$$\frac{\Delta f}{f} = -k_f \Delta t$$

k_f ← Death rate for foxes

$$\frac{\Delta f}{f} = -k_f \Delta t + k_{fm}\, m\, \Delta t$$

$$\Delta f = (-k_f \cdot f + k_{fm} \cdot f \cdot m)\, \Delta t$$

$$\boxed{\begin{aligned} \frac{dm}{dt} &= k_m \cdot m - k_{mf} \cdot mf \\ \frac{df}{dt} &= -k_f \cdot f + k_{fm} \cdot fm \end{aligned}}$$

Solve these given some initial condition for foxes and mice.

$k_m = 2$

$k_{mf} = 0.02$

$k_{fm} = 0.01$

$k_f = 1.06$

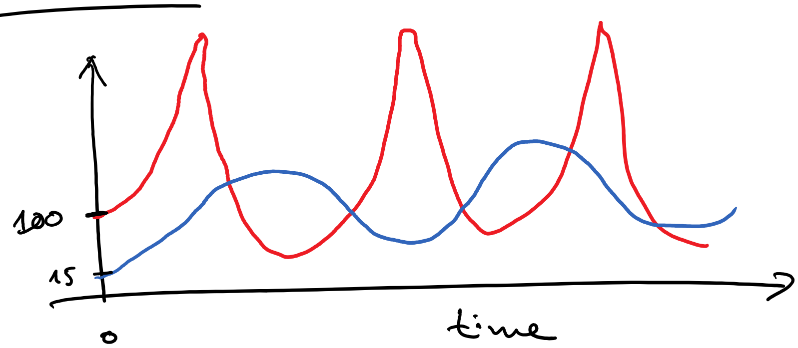
I.C.

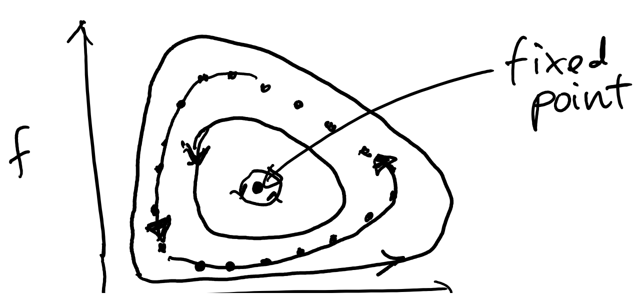
$m(0) = 100$

$f(0) = 15$

$y = \langle m(t), f(t) \rangle$

2 PLOTS:





$\frac{dm}{dt} = 0$

$\frac{df}{dt} = 0$

?

$\frac{d\cdot}{dt} = 0$

PHASE DIAGRAM