Symplectic Integrators

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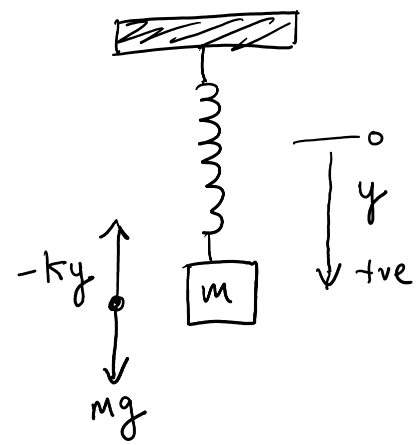
ODE → Runge-Kutta "Black Box"

Symplectic Integrators → try to preserve something or use information about the solution.

e.g. Periodic

Harmonic Oscillator:



$$F = mg - ky$$

to make it easier
$m = 1 \quad k = 1$

$$F = g - y$$ Define $q = 0$ where $F = 0$.

Newton's Laws

$$q = y - g = -F$$

$$F = ma = m\ddot{y} \qquad \dot{q} = \frac{dq}{dt}$$

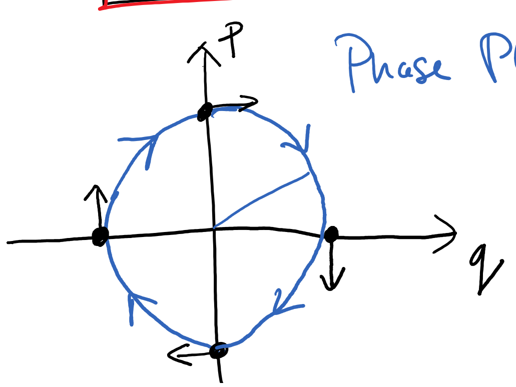
$$\boxed{\ddot{q} = \ddot{y} = F = -q}$$

Momentum $p = mv = m\dot{y} = \dot{q}$

$$\boxed{\begin{aligned} \dot{q} &= p \\ \dot{p} &= -q \end{aligned}}$$

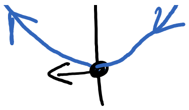
ODE's for the Harmonic Oscillator

Phase Plot for H.O.



$r^2 = p^2 + q^2$
Should be conserved Forever!

$$H = \frac{1}{2}(p^2 + q^2)$$



Hamiltonian of the system

$$H = \tfrac{1}{2}\left(p^2 + q^2\right)$$

Kinetic Energy   Potential Energy

Seperable Hamiltonian

$$\boxed{\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{aligned}}$$

$\frac{1}{2}q^2$

$$\boxed{H = T(p) + U(q)}$$

What happens for the Forward-Euler Method

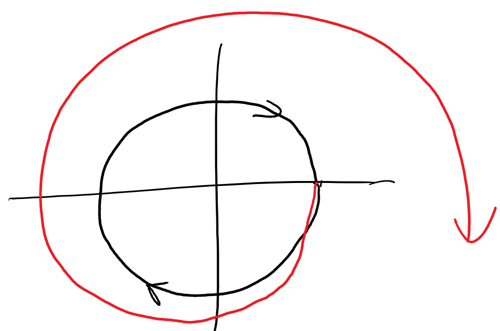
$$q_{n+1} = q_n + h\,p_n$$

$$p_{n+1} = p_n - h\,q_n$$

$$q_{n+1}^2 + p_{n+1}^2 = (1+h^2)(q_n^2 + p_n^2)$$

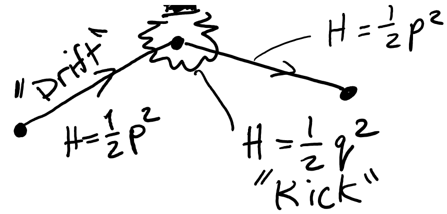
$$r_{n+1}^2 = (1+h^2)\,r_n^2$$

BAD



This will happen to some degree with <u>any</u> black box method.

Leap-Frog Integrator or Strömer-Verlet



For H.O.

$$q_{n+\frac{1}{2}} = q_n + \frac{1}{2} h p_n \qquad \text{"half Drift"}$$

$$p_{n+1} = p_n - h\, q_{n+\frac{1}{2}} \qquad \text{"Kick"}$$

$$q_{n+1} = q_{n+\frac{1}{2}} + \frac{1}{2} h\, p_{n+1} \qquad \text{"final half drift"}$$

$$H = T(p) + U(q)$$

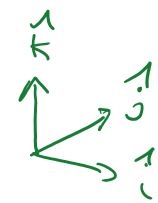
q, $\underline{x}$; p, $\underline{v}$ } vectors

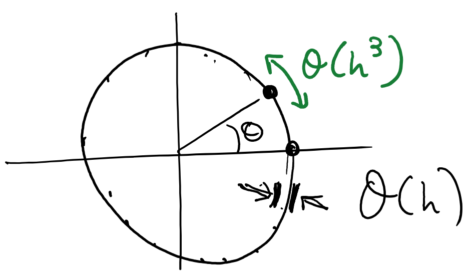
$$\underline{x}_{\frac{1}{2}} = \underline{x}_0 + \frac{1}{2} h \underline{v}_0$$

$$\underline{v}_1 = \underline{v}_0 + h\left(-\nabla U(\underline{x}_{\frac{1}{2}})\right)$$

$$\underline{x}_1 = \underline{x}_{\frac{1}{2}} + \frac{1}{2} h \underline{v}_1$$

$$\nabla U = \frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}$$

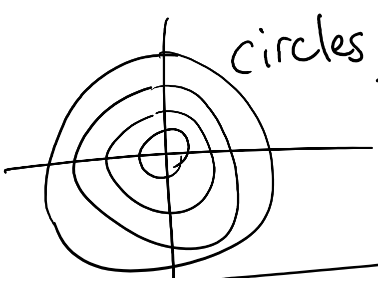




Is this the famous "Free Lunch"?

Perfectly periodic

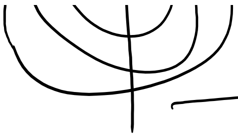
Conserves energy "perfectly" $O(h)$



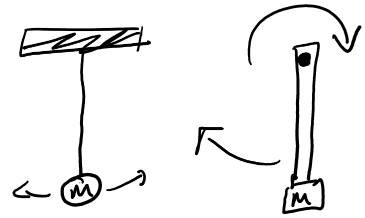
circles!

Accumulated error in angle

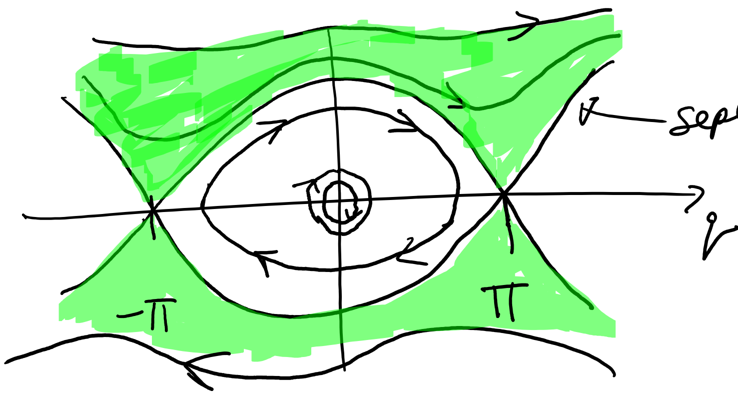
the period is slightly different.



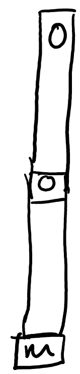
Simple Pendulum



$$H = \frac{1}{2}p^2 - \varepsilon \cos q$$ angle



Deterministic Chaos



$$\underline{X} \equiv \vec{X}$$