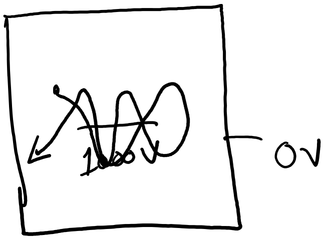
Bi-linear(cubic) Interpolation and Electron Beams

09 November 2020 12:18

$$F = ma$$

ODE

PDE



Electrodynamics

$$\underline{F} = e\left(\underset{\overset{\uparrow}{-\nabla\Phi}}{\underline{E}} + \cancelto{0}{\underline{V}_e \times \underline{B}}\right)$$

$$\underline{a} = \left(\frac{e}{m_e}\right)\left(-\nabla\Phi\right)$$

$$\overset{\dot{v}_x}{\ddot{x}} = \left(\frac{e}{m_e}\right)\left(-\frac{\partial\Phi}{\partial x}\right)$$

$$\ddot{y} = \left(\frac{e}{m_e}\right)\left(-\frac{\partial\Phi}{\partial y}\right)$$

$$\dot{x} = v_x$$

4 equations 1st order

$$e = 1.60 \times 10^{-19}\,C$$

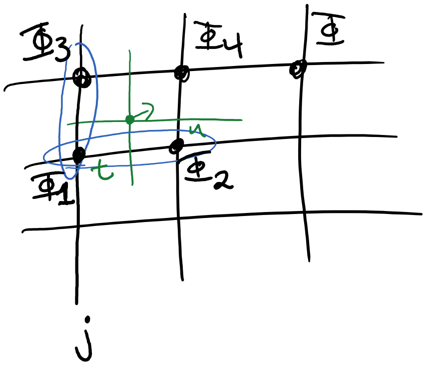
$$m_e = 9.11 \times 10^{-31}\,kg$$

$$\left(\frac{e}{m_e}\right) = 1.76 \times 10^{11}\,\frac{C}{kg}$$

$$\Phi = \left[\frac{Nm}{C}\right] \qquad \nabla\Phi = \left[\frac{N\,\cancel{m}}{C\,\cancel{m}}\right]$$

$$\left(\frac{e}{m_e}\right)\left(-\nabla\Phi\right) = \left[\frac{\cancel{e}}{kg}\right]\left[\frac{N}{\cancel{e}}\right] = \left[\frac{N}{kg}\right] = \left[\frac{\cancel{kg}\,m\,s^{-2}}{\cancel{kg}}\right]$$

$$\ddot{x} = \left[\frac{m}{s^2}\right] \checkmark$$



Interpolation

$$t = \frac{(x - x_j)}{(x_{j+1} - x_j)} = \frac{1}{\Delta}(x - x_j)$$

$$u = \frac{1}{\Delta}(y - y_e)$$

$$\boxed{\frac{\partial t}{\partial x} = \frac{1}{\Delta}}$$

$$\Phi(x, y_\ell) = (1-t)\Phi_1 + t\Phi_2$$

$$\Phi(x_j, y) = (1-u)\Phi_1 + u\Phi_3$$

$$\Phi(x, y_{\ell+1}) = (1-t)\Phi_3 + t\Phi_4$$

$$\boxed{\Phi(x,y) = (1-t)(1-u)\Phi_1 + t(1-u)\Phi_2 + (1-t)u\Phi_3 + tu\Phi_4 \quad \boxed{\forall j,\ell}}$$

Bilinear Interpolation

$$\left.\frac{\partial \Phi}{\partial x}\right|_u = \frac{\partial t}{\partial x}\cdot\frac{\partial \Phi}{\partial t} = \frac{1}{\Delta}\left[-(1-u)\Phi_1 + (1-u)\Phi_2 - u\Phi_3 + u\Phi_4\right]$$

$$= \frac{1}{\Delta}\left[(1-u)\left(\Phi_2-\Phi_1\right) + u\left(\Phi_4-\Phi_3\right)\right]$$

leave $\left.\frac{\partial \Phi}{\partial y}\right|_t = \ ?$ ... you do this!

$$\dot{x} = v_x$$

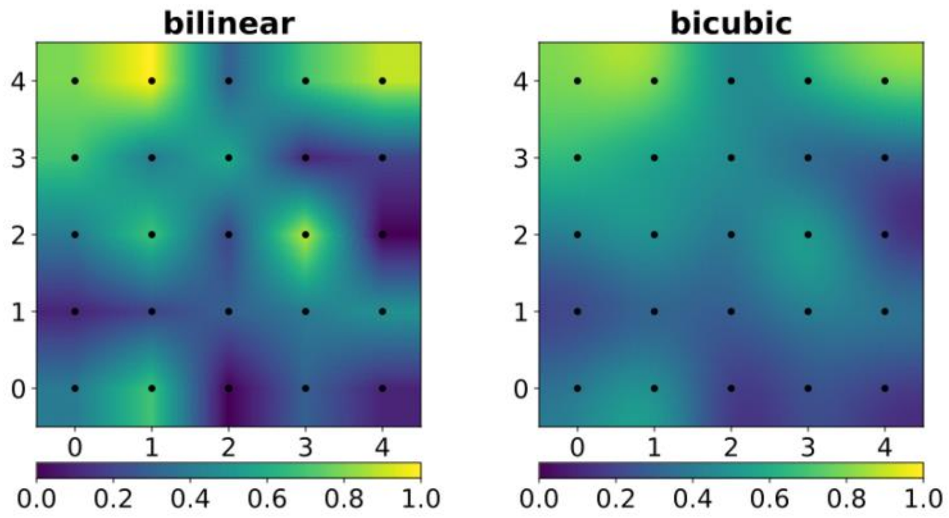
$$\dot{v}_x = \left(\frac{e}{m_e}\right)\left[-\frac{1}{\Delta}\left((1-u)\left(\Phi_2-\Phi_1\right) + u\left(\Phi_4-\Phi_3\right)\right)\right]$$

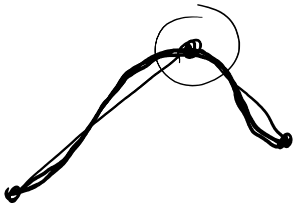
$$\dot{y} = v_y$$

$$\dot{v}_y = \left(\frac{e}{m_e}\right)\left[-\frac{1}{\Delta}\left(\cdots\right)\right]$$

RK 2/4

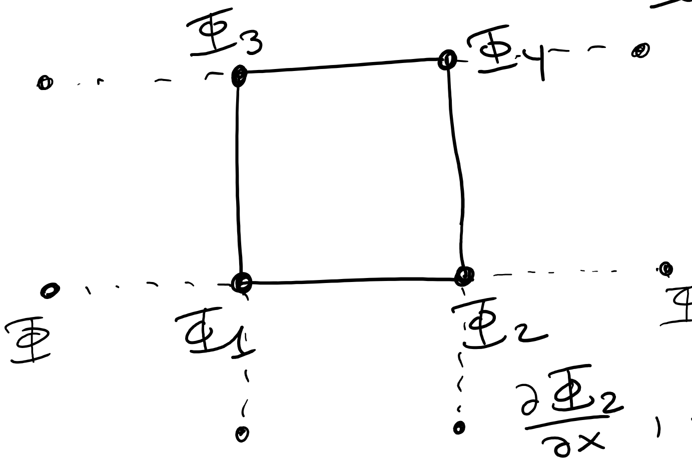
Leapfrog





$$\Phi(t,u) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}\, t^{i} u^{j}$$

16 coefficients



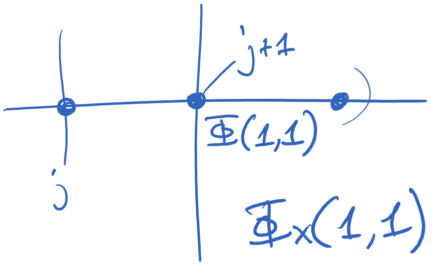
$$\frac{\partial \Phi_2}{\partial x},\ \frac{\partial \Phi_2}{\partial y},\ \frac{\partial \Phi_2}{\partial x \partial y}$$

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & \ddots & \\ a_{20} & & \ddots & \\ a_{30} & & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \Phi(0,0) & \Phi(0,1) & \Phi_y(0,0) & \Phi_y(0,1) \\ \Phi(1,0) & \Phi(1,1) & \Phi_y(1,0) & \Phi_y(1,1) \\ \Phi_x(0,0) & \Phi_x(0,1) & \Phi_{xy}(0,0) & \Phi_{xy}(0,1) \\ \Phi_x(1,0) & \Phi_x(1,1) & \Phi_{xy}(1,0) & \Phi_{xy}(1,1) \end{bmatrix}$$

$\Phi\ (1,1)$

$$\cdot \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \end{bmatrix}$$

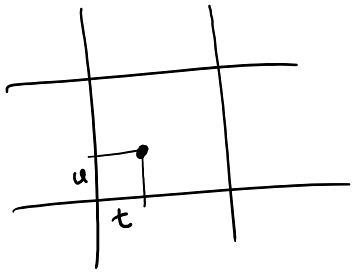
$\Phi_x(1,1)$



$$\cdot \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\Phi_x(1,1) = \frac{1}{2\Delta}\left(\Phi_{j+2,\ell+1} - \Phi_{j,\ell+1}\right)$$

$$\Phi(x,y) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \cdot \begin{bmatrix} a_{ij} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ u \\ u^2 \\ u^3 \end{bmatrix}$$



$t \in [0,1]$

$u \in [0,1]$

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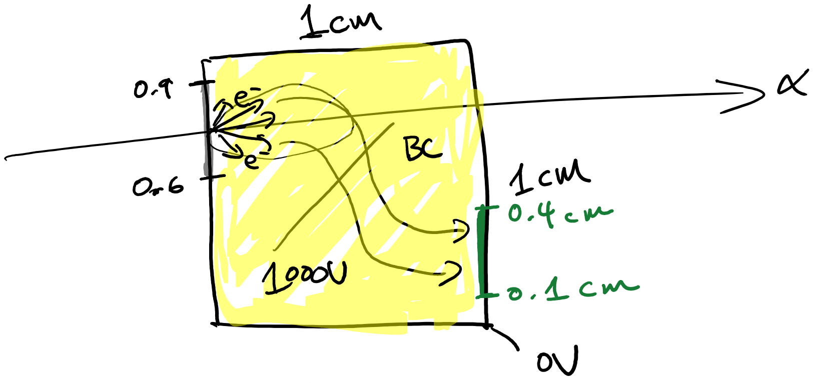
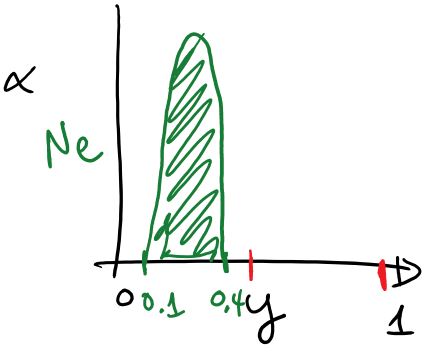
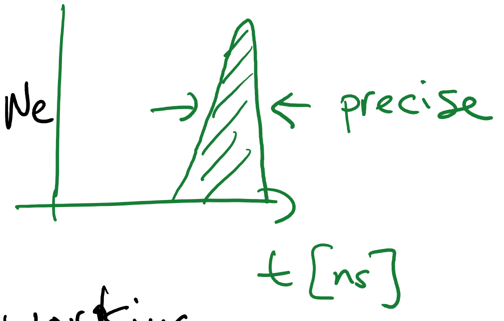
2 Weeks! Exercise





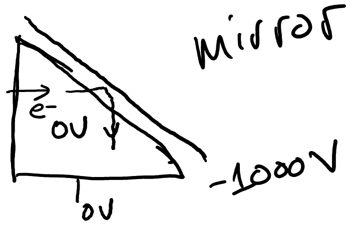
$|v_e| = 10^6$ m/s

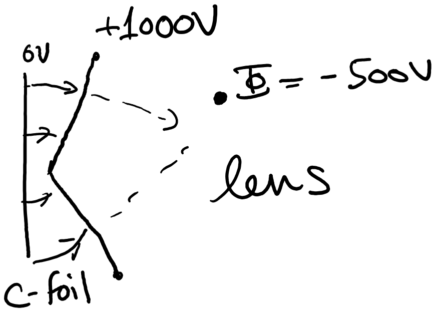
and random angles



1 week to get it working

1 week to come up with a good design.





PRIZE!